CATS COMPETITION[1]

KEY

University of Kentucky High School Math Day
October 2010

NO CALCULATORS, NO CELL PHONES!
WRITE YOUR ANSWERS IN THE PROVIDED BOXES

1 Sam takes his favorite number, multiplies it by 3 and then adds 7 to obtain 40. What is Sam's favorite number?

Answer: 11

2 If the sum of two numbers is 20 and the product is 96, what is the smaller number?

Answer: 8

3 If a rectangle has perimeter 20 inches and area of 24 square inches, what is the length of the shorter side?

Answer: 4

4 Gretchen is five years older than Sam and three years ago Gretchen was twice as old as Sam. How old is Sam today?

Answer: 8

5 The number $n!$ is the product $n(n-1)(n-2)\cdots 1$. How many zeroes are at the end of $15!$?

Answer: 3

6 How many zeroes are at the end of $90!$?

Answer: 21

7 What is the smallest prime factor of 2010?

Answer: 2

[1] CATS stands for CATS Are Top Solvers.

8 What is the largest prime factor of 2010?

Answer: 67

9 Suppose n is a positive integer. The remainder when n is divided by 5 is 1 and the remainder when n is divided by 7 is 2. What is the smallest possible value of n?

Answer: 16

10 According to the standard convention, $1 + 4/2 + 3 = 1 + 2 + 3 = 6$. Including this answer, how many different answers can you obtain by using parentheses to carry out the operations in a different order?

Answer: 4

11 Find two solutions to the equation:

$$\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+x}}} = x.$$

Answer: $-2, 1$

12 Compute the product

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{199}\right).$$

Answer: 100

13 The decimal expansion of N is $0.4444\cdots$. What is the decimal expansion of $\sqrt{N}$?

Answer: $0.6666\ldots$

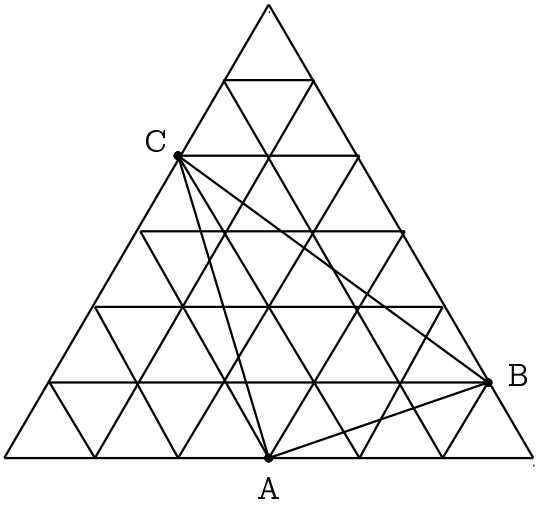
14 Let

$$N = 12345678910111213\ldots9899100.$$

What is the remainder when N is divided by 9?

Answer: 1

15 In the picture, the small triangles are all equilateral and have area 1 square foot. What is the area of $\triangle ABC$?



Answer: 11 ft^2

16 Expand $(1+x)^{12}$ in powers of x:

$$(1+x)^{12} = 1 + 12x + \cdots + x^{12}.$$

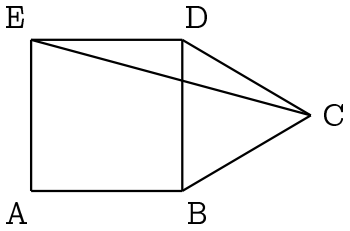
How many terms have even coefficients?

Answer: 9

17 Find the largest 2-digit number A such that A^2 ends with the same two digits as A.

Answer: 76

18 In the picture, $ABDE$ is a square, BCD is an equilateral triangle. Find the measure of $\angle DCE$.



Answer: 15°

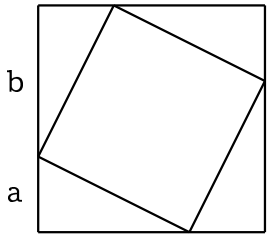
19 Each person in the room shakes hands once with every other person, a total of 136 handshakes. How many people are there in the room?

Answer: 17

20 A rectangular piece of paper, when folded in two, has a rectangular shape similar to the original shape. If the shortest side of the piece of paper (before folding) is 8 inches, what is the longest side?

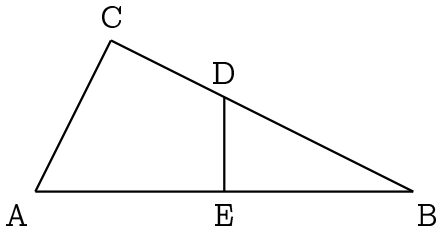
Answer: $8\sqrt{2}$

21 In the picture, the area of the large square is 50% larger than the area of the small square. Assuming that $b > a$, find the ratio b/a.



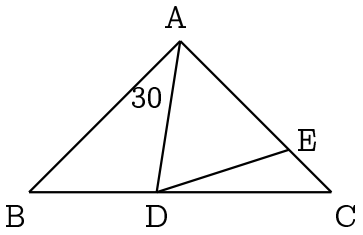
Answer: $2 + \sqrt{3}$

22 In the picture, $\angle ACB$ and $\angle DEB$ are right angles, $|AC| = 12, |CB| = 20$ and $|AE| = |EB|$. Find the area of the quadrilateral $ACDE$.



Answer: $396/5$

23 In the picture, $|AB| = |AC|$, the measure of $\angle BAD$ is 30°, and $|AE| = |AD|$. Find the measure of $\angle EDC$.



Answer: 15°

24 The expansion of $(a+b+c)^3$ is

$$(a+b+c)^3 = a^3+b^3+c^3+3ab^2+3ac^2+3bc^2+3a^2b+3a^2c+3b^2c+6abc$$

and has 10 terms. How many terms does the expansion of $(a+b+c)^{10}$ have?

Answer: 66

25 Let

$$f(x) = -\frac{1}{1+x}.$$

(1) Compute $f(f(f(x)))$.

Answer: x

(2) Compute $f(f(f(f(\cdots f(1)\cdots))))$ (2000 f's).

Answer: -2